Exact Solution of Boundary Layer Flow Due to a Quadratically Shrinking Sheet

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Abstract
Using a Newtonian cooling liquid, the shrinking sheet problem is explored for the closed form solution. The velocity with which the sheet shrinks is assumed to vary as a quadratic function of the distance from the slit. The shrinking sheet is assumed to be permeable and the boundary layer approximation is used. The basic boundary layer equation that governs the flow along the sheet is converted into a set of non-linear ordinary differential equations using suitable similarity transformation. The analytical procedure for solving the resultant system of ordinary differential equations is discussed in detail. The mathematical expressions, involving shrinking parameters, for the stream function and the velocity components are obtained. The linear shrinking sheet problem is shown to be the limiting case of present study. The shrinking sheet problem is shown to be qualitatively much different than the stretching sheet problem though technically it looks like a cosmetic change in one of the boundary conditions.

Keywords: Newtonian Liquid, Quadratically Shrinking Sheet, Suction/Blowing

1. INTRODUCTION
One encounters stretching/shrinking sheet problems often in extrusion processes. For example, polymer extrusion process involves cooling of continuous strips extruded from a die by drawing them into a stationary cooling liquid. The extruded sheet is a useful product in a plastic/glass industries. The properties of the extruded sheet depend on the rate of stretching/shrinking and rate of cooling. The stretching/shrinking of the sheet results in a unidirectional orientation to the extrudate which can improve the mechanical properties of the sheet to a great extent [1]. Crane [2] initiated the study of the stretching sheet problem. He obtained exact solution of the problem using boundary layer approximation. Ever since the pioneering work of Crane [2] on the stretching sheet problem, several works have appeared in the problem by considering various aspects [3-10]. These studies concern linear stretching/shrinking sheet problems handled by analytical/numerical techniques.

However, under realistic situations, the stretching/shrinking of the sheet need not be linear. Also, there can be a situation wherein a variable through flow can exists. With this view point, several works concerning nonlinear stretching/shrinking velocities have appeared in literature. Kumaran and Ramanaiah [11] was the first among others to consider quadratically varying stretching velocity and reported exact solution for the same. Khan and Sanjayanand [12] investigated magnetohydrodynamic boundary layer flow due to a quadratically stretching sheet through viscoelastic fluid saturated porous medium and obtained an analytical solution of the problem relating various physical parameters. Weidman and Magyari [13] solved the stretching sheet problem explicitly assuming an arbitrarily higher degree polynomial for the stretching velocity and concluded that the analytical solutions describe generalized Crane flows whose reciprocal thicknesses always coincide with the negative of their entrainment velocities. Mahesha [10] has numerically investigated the inclined stretching sheet problem with quadratically varying velocity. Closed form similarity solution for viscous flow over an impermeable, quadratically stretching sheet was reported [14]. The study of viscous magnetohydrodynamic flow due to a permeable, quadratically stretched sheet was considered [15]. The work [15] reports exact solution for the mass transfer problem. Recently, analytical solution of magnetohydrodynamic visco-elastic fluids flow is obtained over a sheet shrinking with quadratic velocity [16].

Mention can be made, at this juncture, that the aforementioned studies concerning quadratic stretching/shrinking sheet assumed the similarity solution as indicated by the boundary conditions. More importantly, there is a serious issue of uniqueness of the solution. The exact solutions reported in these studies are of assumed-type. For the first time the exact solution of the stretching sheet problem with a mathematical rigor is obtained, settling the issue of uniqueness of the solution [17]. In the present paper a similar analysis is presented for the shrinking sheet problem with a quadratic shrinking velocity.

2. MATHEMATICAL FORMULATION
Consider steady two-dimensional flow of a viscous incompressible liquid induced by shrinking of a thin elastic sheet. The x-axis of the coordinate system is taken along the sheet and y-axis in the normal direction with origin at the slot. The sheet is assumed to shrink with a speed which varies as a quadratic function of the distance x from the fixed origin. Invoking boundary-layer approximation, the flow along the shrinking sheet is modelled using the following partial differential equations:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}. \quad (2)
\]

The boundary conditions for solving Eqn. (1) and (2) are assumed as follows:
\begin{align}
    u &= -ax - \beta x^2 \quad \text{at} \quad y = 0, \quad (3a) \\
    v &= -\delta x + v_w \quad \text{at} \quad y = 0, \quad (3b) \\
    u &\to 0 \quad \text{as} \quad y \to \infty, \quad (3c)
\end{align}

The condition in Eqn. (3a) is due to noslip condition at the surface, condition (3b) is due to vertical through-flow and condition (3c) is due to ambient quiescent state which implies that the dynamics in the region far away is not disturbed by the shrinking of the sheet. Here we assume that the values of \( \beta \) and \( \delta \) are very small, i.e., \( \beta, \delta \ll 1 \). Thus, the terms involving \( \beta \) and \( \delta \) in Eqn. (3a) and (3b) can be regarded as mild perturbations to a linear shrinking sheet problem.

In order to nondimensionalize the equations and boundary conditions we use the following definition:

\[
  (X,Y) = \left( \frac{a}{\sqrt{V}} (x,y), \frac{(u,v)}{\sqrt{av}} \right),
\]

\[
  \beta^* = \frac{\beta}{\sqrt{\frac{\alpha}{v}}}, \quad \delta^* = \frac{\delta}{2a}, \quad V_w = \frac{v_w}{\sqrt{av}}
\]

Using (4), Eqn. (1) and (2) can be written in the non-dimensional form as follows:

\[
  \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0,
\]

\[
  U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}.
\]

The boundary conditions in (3) take the form:

\[
  U = -X - \beta^* X^2 \quad \text{at} \quad Y = 0, \quad (7a) \\
  V = -2\delta^* X + V_w \quad \text{at} \quad Y = 0, \quad (7b) \\
  U \to 0 \quad \text{as} \quad Y \to \infty, \quad (7c)
\]

To this end, stream function \( \psi(X,Y) \) is introduced so as to satisfy the continuity equation (5), such that

\[
  U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X}.
\]

Substituting Eqn. (8) in (6) we get the following equation for the stream function:

\[
  \frac{\partial^2 \psi}{\partial Y^2} + \frac{\partial \psi \partial^2 \psi}{\partial X \partial Y^2} - \frac{\partial \psi \partial^2 \psi}{\partial Y \partial X \partial Y} = 0.
\]

The boundary conditions to be satisfied by stream function \( \psi \) can be obtained, on using (8) in (7) as:

\[
  \frac{\partial \psi}{\partial Y} = -X - \beta^* X^2 \quad \text{at} \quad Y = 0, \quad (10a) \\
  \frac{\partial \psi}{\partial X} = -2\delta^* X + V_w \quad \text{at} \quad Y = 0, \quad (10b) \\
  \frac{\partial \psi}{\partial Y} \to 0 \quad \text{as} \quad Y \to \infty.
\]

The boundary conditions in Eqn. (10) dictates that the solution of Eqn. (9) may be taken as

\[
  \psi(X,Y) = XF(Y) + \beta^* X^2 g(Y)
\]

Substituting equation (11) into Eqn. (9) and equating the coefficients of \( X, X^2 \) and \( X^3 \), the following set of ordinary differential equations are obtained:

\[
  f''' + ff'' - f'^2 = 0, \quad (11) \\
  g''' + fg'' + 2gf' - 3f'g' = 0, \quad (12) \\
  gg'' - g^2 = 0. \quad (13)
\]

These are three equations for \( f \) and \( g \) in addition to condition (3c) which will be resolved later in the paper. The boundary conditions for solving Eqns. (11) – (13) can be obtained from Eqn. (10) in the form

\[
  f(0) = -V_w, \quad f'(0) = -1, \quad f'(\infty) \to 0, \quad (14) \\
  g(0) = s, \quad g'(0) = -1, \quad g'(\infty) \to 0, \quad (15)
\]

where \( s = \delta^*/\beta^* \). First solving Eqn. (13) for \( g \) subject to the boundary condition (15) gives

\[
  g(Y) = s e^{-\frac{Y}{s}}. \quad (16)
\]

Substituting Eqn. (16) in Eqn. (12) results in the following equation:

\[
  2s^2f''' + 3sf'' + f = \frac{1}{s}. \quad (17)
\]

It is to be noted that Eqn. (17) is linear and of second order. Solving Eqn. (17), subject to boundary conditions in (14), we get:

\[
  f(Y) = \frac{1}{s} + \left(2s + 1 + V_w\right) e^{-\frac{Y}{s}} - 2\left(s + 1 + V_w\right) e^{-\frac{Y}{s}}. \quad (18)
\]

For the solution given in Eqn. (18) to be consistent, it has to satisfy Eqn. (11) and this results in the following consistency condition:

\[
  s^2 + V_w s + 1 = 0. \quad (19)
\]

In order to have the physically realistic exponentially decaying solution it is imperative that \( s \geq 0 \). We therefore take the minimum positive root of Eqn. (19) for our calculations. Using Eqn. (19) in Eqn. (18) yields the solution of Eqn. (11) in the form:

\[
  f(Y) = \frac{1}{s} + s e^{-\frac{Y}{s}}. \quad (20)
\]

In order to compare the present analytical solution with the ones reported in literature we define

\[
  \frac{1}{s} = s^*. \quad (21)
\]

To this end Eqn. (19) is rearranged as

\[
  \frac{1}{s} = -s - V_w. \quad (22)
\]

Equation (20) on using Eqns. (21) and (22) can be written in the form:

\[
  f(Y) = -\frac{1}{s^*} \left(1 - e^{-s^*Y}\right) - V_w. \quad (23)
\]

On using the solutions (16) and (23) along with (21) in (11), the steam function is given by

\[
  \psi(X,Y) = s^* X + \frac{1}{s} e^{-s^*Y} (X + \beta^* X^2). \quad (24)
\]
The expression for the velocity components $U$ and $V$ can be obtained from Eqn. (8) and (24) as:

$$U = -(X + \beta X^2)e^{-s^*Y},$$  \hspace{1cm} (25)

$$V = \frac{1}{s^*} \left(1 - e^{-s^*Y}\right) - 2\beta X e^{-s^*Y},$$  \hspace{1cm} (26)

where,

$$s^* = \frac{-V_w - \sqrt{V_w^2 - 4}}{2}.$$  \hspace{1cm} (27)

We now move on to discuss the results obtained from the exact solution of the considered problem.

3. RESULTS AND DISCUSSION

The mathematical problem considered in the present paper is a generalization of linear shrinking sheet problem with the inclusion of quadratic shrinking wall velocity. One of the major differences between linear and quadratic shrinking sheet problems is that, in the former case the shrinking rate appears implicitly in the solution while as in the latter case the shrinking parameters appear quite explicitly in the solution.

The solutions obtained for the shrinking sheet problem in the earlier studies [11-16] in the literature are of “assumed” type to conform to the requirement of the boundary condition where as the one obtained in the present paper involves mathematical rigor [17]. This is one of the salient features of the present paper.

It should be noted that $V_w > 0$ represents blowing and $V_w < 0$ corresponds to suction. Also, from equation (27) it is clear that the solution exists for $V_w \leq -2$. In this domain, $s^*$ is an increasing function of $V_w$ as can be seen from Fig. 1.

![Fig. 1 Variation of $s^*$ with $V_w$](image1)

One can obtain the streamlines from Eqn. (24) by setting $\psi(X, Y) = C$, where $C$ is a constant. This results in the following equation for stream lines

$$Y = -\frac{1}{s^*} \log \left(\frac{s^* (C - s^* X)}{X + \beta X^2}\right).$$

Figure 2 shows the stream lines with $\beta^* = 0.01$ and $V_w = -2$. It is quite obvious that the steam lines corresponding to shrinking sheet are exactly opposite to those observed in stretching sheet problem, due to the assumed boundary conditions.

![Fig. 2 Streamlines $\psi(X, Y) = C$ with $\beta^* = 0.01$, $V_w = -2$](image2)

Figure 3 is a plot of various stream lines setting $\psi(X, Y) = 3$ for different values of shrinking parameter $\beta^*$. It is clear from Figure 3 that there is a lifting effect in case of quadratic shrinking $\beta^* > 0$ case.

![Fig. 3 Streamlines $\psi(X, Y) = 3$ for different values of $\beta^*$ with $V_w = -2$](image3)

Figure 4 depicts stream lines $\psi(X, Y) = 1$ for different values of $V_w$. It is evident from the figure that increasing suction in quadratic shrinking sheet problem results in decreasing the lifting effect as we go downstream. This is quite opposite to what is observed in case of stretching sheet problem.

![Fig. 4 Streamlines $\psi(X, Y) = 1$ for different values of $V_w$ with $\beta^* = 0.01$](image4)
4. CONCLUSIONS

An exact solution of the quadratic shrinking sheet problem valid for some regime of suction parameter is obtained with mathematical rigor. In the quadratic shrinking sheet problem there is lifting of the liquid as we go downstream. The lifting effect is reduced with suction parameter.

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