Existence and Uniqueness of Solution to the Direct Electromagnetic Scattering Problem with Leontovich Boundary Condition

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Abstract

This paper investigate scattering of time harmonic electromagnetic plane waves by an obstacle with the most general impedance boundary condition known as the Leontovich boundary condition. Here it prove existence and uniqueness of the solution to the forward scattering problem when the scattering data is available for a set of frequencies but only for one incident direction.. Further it is proved that the solution is analytic in a complex neighborhood of a real frequency.

Key Words: Leontovich Boundary Condition, Scattering, Time Harmonic Electromagnetic Plane Waves

1. INTRODUCTION

Differential equations are necessary for a mathematical model of physical phenomena. Many of the general laws of nature in physics, chemistry and engineering are naturally expressed in the language of differential equations. At the beginning, mathematicians were mostly interested in modelling natural phenomenon using differential equations and solving them but they did not consider the existence and uniqueness of solutions. As the mathematical model's complexity grew, solving the resulting differential equations became more complex, especially in the case of non-linear differential equations. Hence mathematicians started to investigate approximate solutions of differential equations which gave rise to the new field of mathematics namely, the numerical solution of differential equations. This gave rise to the question of existence and uniqueness of solution to differential equations. At the beginning of 20th century Hadamard defined that for a problem to well-posed [1] the following conditions should hold:

1. Existence of the solution
2. Uniqueness of the solution
3. Continuous dependence of the solution on the given data

Let us consider the scattering of time harmonic electromagnetic waves by an obstacle D in $\mathbb{R}^3$in homogenous media. We assume that the boundary of the obstacle $\partial\mathrm{D}$ is of class $\mathrm{C}^2(\partial\mathrm{D})$ and let $\mathrm{D_e} = (\mathbb{R}^3\backslash\overline{\mathrm{D}})$ be connected. We consider the exterior boundary value problem for the Maxwell's System

$$\begin{cases} \mathrm{curl}\boldsymbol{E} - ik\boldsymbol{H} = 0 \ \text{ in } \ D_e \\ \mathrm{curl}\boldsymbol{H} + ik\boldsymbol{E} = 0 \ \text{ in } \ D_e \end{cases} \quad (1)$$

with the most general impedance boundary condition known as the Leontovich boundary condition

$$\nu \times \boldsymbol{H} - \lambda(\nu \times \boldsymbol{E}) \times \nu = 0 \text{on} \partial D, \quad (2)$$

$\lambda \geq 0,\ \lambda \in \mathrm{C}^1(\partial\mathrm{D}),\ \nu$ is the outward normal and the scattered fields $\mathbf{E}^{\mathrm{s}}$, $\mathbf{H}^{\mathrm{s}}$ satisfying the Silver-Müller radiation conditions

$$\begin{cases} \lim\limits_{r\to\infty}(\boldsymbol{H}^s \times x - r\boldsymbol{E}^s) = 0, \\ \lim\limits_{r\to\infty}(\boldsymbol{E}^s \times x + r\boldsymbol{H}^s) = 0, \end{cases} \quad (3)$$

and from [2] we know that the Silver-Müller radiation conditions are equivalent to the Sommerfeld radiation condition for the Cartesian components,

$$\begin{cases} \lim\limits_{r\to\infty} r\left(\frac{\partial \boldsymbol{E}^s}{\partial r} - ik\boldsymbol{E}^s\ \right) = 0, \\ \lim\limits_{r\to\infty} r\left(\frac{\partial \boldsymbol{H}^s}{\partial r} - ik\boldsymbol{H}^s\ \right) = 0. \end{cases} \quad (4)$$

Also, the scattered fields have the following asymptotic behavior [2]

$$\begin{cases} \boldsymbol{E}^s(x) = \frac{e^{ikr}}{r\tilde{A}—}\left\{\boldsymbol{E}_\infty(\hat{x}) + O\left(\frac{1}{r\tilde{A}—}\right)\right\}, \quad r \to \infty, \\ \boldsymbol{H}^s(x) = \frac{e^{ikr}}{r\tilde{A}—}\left\{\boldsymbol{H}_\infty(\hat{x}) + O\left(\frac{1}{r\tilde{A}—}\right)\right\}, \quad r \to \infty, \end{cases} \quad (5)$$

where $\mathbf{E}_\infty$ and $\mathbf{H}_\infty$ are known as the far field pattern or the scattering amplitude. Here $= \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathbf{s}}$, and $\mathbf{H} = \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathbf{s}}$, are the incident electric and magnetic fields.

In the direct problem we are looking for the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ in the space $\mathbf{H}^2\left(\mathrm{B}_\rho\backslash\overline{\mathrm{D}}\right)$ for some $\rho > \rho_0 > 0$, $\overline{\mathrm{D}} \subset \mathrm{B}_\rho$, where $\mathbf{H}(\mathrm{D})$ is $\left(\mathrm{H}(\mathrm{D})\right)^3$the three dimensional product of standard Sobolev space. Also, $\mathbf{E}$, $\mathbf{H} \in \mathrm{C}^\infty(\mathbb{R}^3\backslash\mathrm{D})$ with $\mathbf{E}^{\mathbf{s}}$, $\mathbf{H}^{\mathbf{s}}$ satisfying the Silver-Müller radiation condition.

This scattering problem is one of most realistic mathematical models of electromagnetic prospecting, especially in radar . Both direct and inverse problems are of current research interest. The direct scattering problem does not satisfy the conditions of general theorems in [3] applying Lax-Phillips scattering theory for dissipative systems to the Maxwell system [4]. For available results we refer to the paper [5], where there is a solvability theory for the direct scattering problem in $\mathbf{H}^1$ spaces. Also by using the method of singular solutions of one of the authors [6] it was proven that the scattering data at fixed frequency k and all directions of incident waves and receivers uniquely determine D and λ on ∂D. In many practical situations the complete scattering data at a fixed frequency are not available, and what is available are data at a fixed incident direction and at all (or many) frequencies, k. Uniqueness of polyhedral D and of the Leontovich boundary condition from the scattering data at one frequency and one incident direction is already known [7].

The main purpose of this paper is to show the analyticity, of the solution to the Maxwell system with the boundary condition (2), with respect to k. For acoustical (scalar) case analyticity follows directly from the Lax-Phillips results. But for the Maxwell system with the boundary condition (2) analyticity was not known, and a substantial part of our paper is devoted to its proof. For this purpose we reduce the boundary value problem (1), (2) to an elliptic boundary value problem for vector Helmholtz equation and to an application of the Lax-Phillips method of studying scattering problems by using elliptic problems in bounded domains. By using this reduction one can transfer to the electromagnetic scattering problem most of the elliptic theory, including solvability in L^p and Hölder spaces and most likely in Sobolev spaces of negative order. For our goals an elliptic problem is crucial to use a simple but useful statement that an analytical perturbation of an invertible operator between Banach spaces which do not depend on k has the inverse, analytic with respect to k. This result is important for the inverse problem which is to prove uniqueness of the obstacle D and that of λ on D.

2. ELLIPTIC PROBLEM IN A BOUNDED DOMAIN

It is well known that the scattering problem for Maxwell's System (1), (2) and (3) is equivalent to the following scattering problem for vector Helmholtz equation [2,3,8]

$$\begin{cases} \Delta \boldsymbol{E} + k^2 \boldsymbol{E} = 0 & \text{in } D_e, \\ \text{div}\boldsymbol{E} = 0 & \text{on } \partial D, \\ \nu \times \text{curl}\boldsymbol{E}^s - ik\lambda(\nu \times \boldsymbol{E}^s) \times \nu = \boldsymbol{g}_\tau & \text{on } \partial D, \\ \lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{E}^s}{\partial r} - ik\boldsymbol{E}^s\right) = 0, & \end{cases} \tag{6}$$

and the equation

$$\boldsymbol{H} = -\frac{i}{k}\text{curl}\boldsymbol{E}, \tag{7}$$

The boundary data is given by $\mathbf{g}_\tau = -\nu \times \text{curl}\,\mathbf{E}^i + ik\lambda(\nu \times \mathbf{E}^i) \times \nu$.

Let us consider the following boundary value problem for the vector Helmholtz equation

$$\begin{aligned} &\Delta \boldsymbol{v} + k^2 \boldsymbol{v} = \boldsymbol{f} && \text{in}\Omega = B_R \backslash \overline{D}, \\ &\left.\begin{matrix} \nu \times \text{curl}\boldsymbol{v} - ik\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div}\boldsymbol{v} \end{matrix}\right\} = \boldsymbol{g} && \text{on}\partial D, \\ &\boldsymbol{v} = \boldsymbol{g}_0 && \text{on}\partial B_R, \end{aligned} \tag{8}$$

$\in \mathbf{H}^2(\Omega), \mathbf{f} \in \mathbf{H}^0(\Omega), \mathbf{g} \in \mathbf{H}_\tau^{\frac{1}{2}}(\partial \mathrm{D})$ and $\mathbf{g}_0 \in \mathbf{H}^{\frac{3}{2}}(\partial \mathrm{B_R})$.

Using elementary concepts [1,9,10], it can be shown that the boundary value problem (8) is elliptic. Let $\mathcal{A}(\mathrm{k})\colon \mathbf{H}^2(\Omega) \to \mathbf{H}^0(\Omega) \times \mathbf{H}_\tau^{\frac{1}{2}}(\partial \mathrm{D}) \times \mathbf{H}^{\frac{3}{2}}(\partial \mathrm{B_R})$ be the operator corresponding to the elliptic boundary value problem (8). Since the boundary value problem (8) is elliptic we have that the operator $\mathcal{A}(\mathrm{k})$ is Fredholm.

For an operator $\mathcal{T} \in \mathrm{C}(\mathrm{X}, \mathrm{Y})$, where X and Y are Banach spaces, to be Fredholm means that,

- The kernel Ker $\mathcal{T} = \{x \in X : \mathcal{T}x = 0\}$ is finite dimensional,
- The range $R(\mathcal{T})$ is closed in Y,
- The cokernel Coker $\mathcal{T} = Y/R(\mathcal{T})$ is finite dimensional.

The index of the Fredholm operator is defined as,

$$\text{ind}\mathcal{T} = \text{dimKer}\,\mathcal{T} - \text{dimCoker}\,\mathcal{T}.$$

To prove the invertibility of the operator $\mathcal{A}$, we have to prove that the index $\text{ind}\mathcal{A} = 0$. The following theorem [1], implies that the index is insensitive to small perturbations of the Fredholm operator.

Theorem 2.1 *Let $\mathcal{T}, \mathcal{T}_1 \in C(X, Y)$ and let $\mathcal{T}$ be Fredholm.*

Then there exists a $\delta > 0$ (depending on $\mathcal{T}$ such that $\| \mathcal{T} - \mathcal{T}_1 \| < \delta$ implies that $\mathcal{T}_1$ is Fredholm and $\text{ind}\mathcal{T}_1 = \text{ind}\mathcal{T}$.

The following result, due to Atkinson [1], Theorem 1.9, p. 370, will be needed to study solvability of the boundary value problem (8) and analyticity of its solution and of the solutions of the scattering problem with respect to k.

Theorem 2.2 *Let $\mathcal{T}(\kappa)$ be a family of compact operators in X holomorphic for $\kappa \in \mathcal{C}$. Call κ a singular point if 1 is an eigenvalue of $T(\kappa)$. Then either all $\kappa \in \mathbb{C}$ are singular points or there are only a finite number of singular points in each compact subset of $\mathbb{C}$.*

From [1] required the following lemma.

Lemma 2.1 *If $\mathcal{T}(\kappa) \in C(X, Y)$ is holomorphic with respect to k at k_0 and $\mathcal{T}(\kappa_0)^{-1} \in C(Y, X)$ does exists, then $\mathcal{T}(\kappa)^{-1}$ exists, belongs to $C(Y, X)$ and is holomorphic with respect to k for sufficiently small $|\kappa - \kappa_0|$.*

We recall that the operator $\mathcal{A}(k)$ maps the solution of the boundary value problem (8) from $\boldsymbol{H}^2(\Omega)$ into $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}^{\frac{1}{2}}(\partial D) \times \boldsymbol{H}^{\frac{3}{2}}(\partial B_R)$. Let $k \in \mathbb{C}\backslash S$ and $\boldsymbol{V}(k)$ be the restriction of the inverse $\mathcal{A}^{-1}(k)$ onto $\boldsymbol{H}^0(\Omega) \times \{0\} \times \{0\}$ which is considered as a mapping from $\boldsymbol{H}^0(\Omega)$ into $\boldsymbol{H}^2(\Omega)$.

Lemma 2.2 *The map $\boldsymbol{V}(k, :)\colon \boldsymbol{H}^0(\Omega) \to \boldsymbol{H}^2(\Omega)$ is analytic with respect to frequency $k \in \mathbb{C}\backslash S$.*

Proof. The operator $\mathcal{A}(k)$ is analytic with respect to k. From lemma (2.1) we have that its inverse is also analytic and hence its restriction $\boldsymbol{V}(k)$ is holomorphic on $\mathbb{C}\backslash S$.

Theorem 2.3 *There is a set $S \subset \mathbb{C}$ without accumulation points in $\mathbb{C}$ such that for $k \in \mathbb{C}\backslash S$ there is an unique solution to the boundary value problem (8) in $\boldsymbol{H}^2(\Omega)$.*

Proof. Let $k_0 = -ik_1$ for some real $k_1 \neq 0$. We now show the uniqueness of the solution to the following elliptic boundary value problem (8). Letting in (8) $f = 0, g = 0$, multiplying by $\overline{\boldsymbol{v}}$, and integrating by parts we have

$$\int_{\Omega} |\Delta \boldsymbol{v}|^2 + k^2|\boldsymbol{v}|^2 + k_1\lambda \int_{\partial D} |\boldsymbol{v}_T|^2 = 0,$$

where $\boldsymbol{v}_T$ is a tangential component of $\boldsymbol{v}$. Therefore, $\boldsymbol{v} = 0$. Hence we have uniqueness when $k = k_0$.

Now we show that the index of the operator $\mathcal{A}$ is zero. Let $\mathcal{A}_\theta, 0 \leq \theta \leq 1$, be the similar operator corresponding to the following boundary value problem:

$$\begin{gathered} \Delta \boldsymbol{v} = \boldsymbol{f} \text{ in } \Omega, \\ \partial_3 v_1 - \partial_1 v_3 = g_1 \\ \partial_3 v_2 - \partial_2 v_3 = g_2, \\ \theta\partial_1 v_1 + \theta\partial_2 v_2 + \partial_3 v_3 = g_0 \text{ on } \partial D, \\ \boldsymbol{v} = \boldsymbol{g}_0 \text{ on } \partial B_R. \end{gathered} \tag{9}$$

We defined the boundary condition at a point $x_0 \in \partial D$ in the coordinates used to check the ellipticity of the boundary value problem (8) above. It is not hard to check that this definition does not depend on the choice of an orthonormal base in the tangent plane to ∂D at x_0 and that is can be written as a first order linear partial differential operator with $C^1(\partial D)$ coefficients. As (8), the boundary value problem (9) is elliptic and hence the operator $\mathcal{A}_\theta$ is Fredholm. Notice that $\mathcal{A}_\theta$ is continuous with respect to θ (in the standard operator norm). Therefore, ind $\mathcal{A}_\theta$ is constant when $0 \leq \theta \leq 1$. $\mathcal{A}_0$ corresponds to the mixed Dirichlet Neumann boundary value problem which is uniquely solvable. Hence the index of this boundary value problem is zero, i.e. ind $\mathcal{A}_0 = 0$, which implies that ind $\mathcal{A}_1 = 0$. Since the ellipticity is determined only by principal parts, the operators $\mathcal{A}_\theta(k), 0 \leq \theta \leq 1$, are Fredholm, continuously depend on θ, and we similarly have ind $\mathcal{A}(k)$ = ind $\mathcal{A}(0)$= ind $\mathcal{A}_0 = 0$.

Now due to uniqueness, the elliptic boundary value problem (8) is uniquely solvable for $k = k_0$.

To complete the proof for any $k \in \mathbb{C}$ we write the boundary value problem (8) as a compact perturbation of the boundary value problem (8) with $k = k_0$:

$$\begin{aligned} &\Delta \boldsymbol{v} + k_0^2\boldsymbol{v} - (k_0^2 - k^2)\boldsymbol{v} = \boldsymbol{f} && \text{in}\Omega, \\ &\left.\begin{aligned}\nu \times \text{curl}\boldsymbol{v} - ik_0\lambda(\nu \times \boldsymbol{v}) \times \nu + i(k_0^2 - k^2)\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div}\boldsymbol{v}\end{aligned}\right\} = \boldsymbol{g} && \text{on}\partial D, \\ &\boldsymbol{v} = \boldsymbol{g}_0 && \text{on}\partial B_R. \end{aligned}$$

We introduce the operator $\mathcal{B}(k) = \mathcal{A}(k) - \mathcal{A}(k_0)$. Then (8) can be written as $(\mathcal{A}(k_0) + \mathcal{B}(k))\boldsymbol{v} = (\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{g}_0)$, or

$$\boldsymbol{v} + \mathcal{A}(k_0)^{-1}\mathcal{B}(k)\boldsymbol{v} = \mathcal{A}(k_0)^{-1}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{g}_0) \tag{10}$$

Observe that

$$\mathcal{B}(k)(\boldsymbol{v}) = ((k^2 - k_0^2)\boldsymbol{v}, (i(k_0^2 - k^2)\lambda(\nu \times \boldsymbol{v}) \times \boldsymbol{v}, 0), 0).$$

Hence $\mathcal{B}$ is a compact operator from $\boldsymbol{H}^2(\Omega)$ into $\boldsymbol{H}^0(\Omega) \times \boldsymbol{H}_\tau^{\frac{1}{2}}(\partial D) \times \boldsymbol{H}^{\frac{3}{2}}(\partial B_R)$ and hence $\mathcal{A}(k_0)^{-1}\mathcal{B}(k): \boldsymbol{H}^2 \to \boldsymbol{H}^2$ is compact and obviously holomorphic with respect to $k \in \mathbb{C}$. Since $k = k_0$ is not a singular point of $\mathcal{A}(k_0)^{-1}\mathcal{B}(k)$, the set S of its singular points k is discrete. Hence the equation (10) and therefore the boundary value problem (8) are uniquely solvable when $k \in \mathbb{C}\backslash S$.

3. EXISTENCE, UNIQUENESS AND ANALYTICITY

Theorem 3.1 If $k \in \mathbb{R}$ and $k \neq 0$ then there exists a unique solution to the scattering problem (2.1). Moreover, this solution can be holomorphically (with respect to k) extended in a neighborhood of $k_1 \in \mathbb{R}\backslash\{0\}$.

Proof. To prove solvability and analyticity for the scattering problem (2.1) we use the Lax-Phillips method. To do so we first replace the problem by another scattering problem which is more suitable for this method.

Let $\boldsymbol{E}^* \in \boldsymbol{H}^2(\Omega)$ solve the following elliptic boundary value problem.

$$\begin{aligned} &\Delta \boldsymbol{E}^* + k^2\boldsymbol{E}^* = 0 && \text{in}\Omega, \\ &\left.\begin{aligned}\nu \times \text{curl}\boldsymbol{E}^* - ik\lambda(\nu \times \boldsymbol{E}^*) \times \nu \\ \text{div}\boldsymbol{E}^*\end{aligned}\right\} = \boldsymbol{g} && \text{on}\partial D, \\ &\boldsymbol{E}^* = 0 && \text{on}\partial B_R. \end{aligned} \tag{11}$$

We can choose R so that this elliptic problem is uniquely solvable. Consider domain Ω_0 containing $\overline{D}$ such that $\overline{\Omega}_0$ lies in B_R. Let ϕ be a C^∞cutoff function that is 1 near $\overline{D}$ and 0 in $B_R\backslash\Omega_0$. Then $\boldsymbol{E}_* = \boldsymbol{E} - \phi\boldsymbol{E}^*$, $\boldsymbol{E}_* \in \boldsymbol{H}^2(B_\rho\backslash\overline{D})$, solves the following scattering problem

$$\begin{gathered} \Delta \boldsymbol{E}_* + k^2\boldsymbol{E}_* = \boldsymbol{f}_* \quad \text{in } D_e, \\ \lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{E^s}_*}{\partial r} - ik\boldsymbol{E^s}_*\right) = 0, \end{gathered} \tag{12}$$

where $\boldsymbol{f}_* = \boldsymbol{f} - (\Delta + k^2)(\phi\boldsymbol{E}^*) \in \boldsymbol{H}^0(\Omega)$. Conversely, (12) implies (6).

To show unique solvability of (12) we first prove uniqueness, i.e. that the solution $\boldsymbol{E}$ to the homogeneous scattering problem

$$\begin{aligned} &\Delta \boldsymbol{E} + k^2\boldsymbol{E} = 0 && \text{in}D_e, \\ &\text{div}\boldsymbol{E} = 0 && \text{on}\partial D, \\ &\nu \times \text{curl}\boldsymbol{E} - ik\lambda(\nu \times \boldsymbol{E}) \times \nu = 0 && \text{on}\partial D, \\ &\lim_{r\to\infty} r\left(\frac{\partial \boldsymbol{E}}{\partial r} - ik\boldsymbol{E}\right) = 0, \end{aligned} \tag{13}$$

is identically zero. Since $div\boldsymbol{E} = 0$, $-(\Delta + k^2)\boldsymbol{E} = \text{curl curl}\,\boldsymbol{E} - k^2\boldsymbol{E}$. Multiplying this by $\overline{\boldsymbol{E}}$ and integrating by parts we obtain

$$\int_{B_\rho\backslash\overline{D}} (|\text{curl}\boldsymbol{E}|^2 - k^2|\boldsymbol{E}|^2)dx + ik\int_{S_\rho} (\nu \times \overline{\boldsymbol{E}}) \cdot \boldsymbol{H}ds + ik\lambda\int_{\partial D} |\boldsymbol{E}_T|^2 ds = 0. \tag{14}$$

Using the scattering data for non-zero real k, by taking the imaginary part of the above equation we have

$$Re\int_{S_\rho} (\nu \times \overline{\boldsymbol{E}}) \cdot \boldsymbol{H}ds = -\lambda\int_{\partial D} |\boldsymbol{E}_T|^2 ds \leq 0. \tag{15}$$

Hence uniqueness follows from [2].

Now we look for a solution $\boldsymbol{E}_*$ to (12) which is of the form

$$\boldsymbol{E}_* = \boldsymbol{w} - \phi(\boldsymbol{w} - \boldsymbol{v}) \tag{16}$$

where $\boldsymbol{v}(:, \boldsymbol{f}^*)$ is a solution to the following elliptic boundary value problem

$$\begin{aligned} &\Delta \boldsymbol{v} + k^2\boldsymbol{v} = \boldsymbol{f}^* && \text{in}\Omega, \\ &\left.\begin{aligned}\nu \times \text{curl}\boldsymbol{v} - ik\lambda(\nu \times \boldsymbol{v}) \times \nu \\ \text{div}\boldsymbol{v}\end{aligned}\right\} = \boldsymbol{g} && \text{on}\partial D, \\ &\boldsymbol{v} = 0 && \text{on}\partial B_R, \end{aligned} \tag{17}$$

where $\boldsymbol{f}^* \in \boldsymbol{H}^0(B_R)$, $\boldsymbol{f}^* = 0$ in D and $\boldsymbol{w}(:,\boldsymbol{f}^*)$ is a solution to the Helmholtz equation in free space

$$\Delta \boldsymbol{w} + k^2 \boldsymbol{w} = \boldsymbol{f}^*,$$

which satisfies the radiation condition. Hence

$$\begin{aligned}\Delta \boldsymbol{E}_* + k^2 \boldsymbol{E}_* &= \Delta \boldsymbol{w} + k^2 \boldsymbol{w} - \Delta\phi(\boldsymbol{w} - \boldsymbol{v}) \\ &-2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v}) - \phi(\Delta(\boldsymbol{w} - \boldsymbol{v}) + k^2(\boldsymbol{w} - \boldsymbol{v})) \\ &= \boldsymbol{f}^* + K\boldsymbol{f}^*,\end{aligned}$$

where $K\boldsymbol{f}^* = -\Delta\phi(\boldsymbol{w} - \boldsymbol{v}) - 2\nabla\phi \cdot \nabla(\boldsymbol{w} - \boldsymbol{v})$ in $\Omega_0 \backslash D$. $\boldsymbol{E}_*$ solves the equation $\Delta \boldsymbol{E}_* + k^2 \boldsymbol{E}_* = \boldsymbol{f}_*$ on Ω if and only if $\boldsymbol{f}^*$ solves the equation

$$\boldsymbol{f}_* = \boldsymbol{f}^* + K\boldsymbol{f}^*. \quad (18)$$

Since $\boldsymbol{v}(;\boldsymbol{f}^*), \boldsymbol{w}(;\boldsymbol{f}^*)$ are continuous mappings from $\boldsymbol{H}^0(\Omega)$ into $\boldsymbol{H}^2(\Omega)$, the operator K is compact from $\boldsymbol{H}^0(\Omega)$ into itself. Thus equation (18) is Fredholm and hence uniqueness of its solutions implies implies solvability. To show uniqueness we let $\boldsymbol{f}_* = 0$, then $\boldsymbol{E}_*$ is a solution to the homogeneous scattering problem and therefore $\boldsymbol{E}_* = 0$, which implies that $\boldsymbol{w} = \phi(\boldsymbol{w} - \boldsymbol{v})$ on Ω. Since $\phi = 1$ near ∂D, it follows that $\boldsymbol{v} = 0$ on Ω near ∂D and we can extend $\boldsymbol{v}$ as zero onto D, so that the extended $\boldsymbol{v} \in \boldsymbol{H}^2(B_R)$. Now from the equations for $\boldsymbol{w}$ and $\boldsymbol{v}$ one can notice that $\boldsymbol{w} - \boldsymbol{v}$ solves the homogeneous Helmholtz equation in B_R, also since $\phi = 0$ near ∂B_R, from $\boldsymbol{w} = \phi(\boldsymbol{w} - \boldsymbol{v})$ we have $\boldsymbol{w} = 0$ on ∂B_R. From (17) it follows that $\boldsymbol{v} = 0$ on ∂B_R. Hence $\boldsymbol{w} - \boldsymbol{v}$ solves the homogeneous Dirichlet problem for the Helmholtz equation on $B_R = 0$ and due to our assumption on B_R we yield $\boldsymbol{w} - \boldsymbol{v} = 0$ on B_R. Therefore $\boldsymbol{w} = \phi(\boldsymbol{w} - \boldsymbol{v}) = 0$ on B_R, which implies that $\boldsymbol{f}^* = 0$.

4. CONCLUSION

We have proved the existence and uniqueness of solution to the Maxwell's system with the most general impedance boundary condition known as the Leontovich boundary condition. Most importantly we have established that the scattering solution depends analytically on the frequency, k. The interesting question is that of the inverse problem, can we uniquely determine the scatterer and also the coefficient λ when scattering data is available for only one incident direction and for different frequencies. To prove these results for the inverse problem, the fact that the solution depends analytically on the frequency will be useful.

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